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SOME REMARKS ON APPROXIMATE COMPUTATION.

By M. J. Babb.

It has generally happened in the past that teachers of elementary mathematics have had little or no acquaintance with what is generally called the higher mathematics. Even when a boy went to college he promptly forgot his elementary mathematics, and if by chance he became a teacher in the secondary schools there was no one to advise with him on means of applying what he had learned to the work in hand. Necessarily he drifted into the stereotyped manner of teaching the subject and his college work was only a pleasant memory. The college teachers were often recruited from men who naturally had very little knowledge of the problems of elementary education. Very few would have been able to grasp the point of view of the youthful mind; many were thoroughly ingrossed in their subject and had found ideals of rigidity of statement and proof as far beyond the ken of the elementary schools as they were in advance of the notions of those who intuitively founded the subject wiser than they knew.

In this paper I am privileged to address a body of men and women who are banded together, not as college teachers, not as teachers of arithmetic or algebra or geometry, but as teachers of mathematics. The spirit of "put yourself in his place" is growing. The normal schools owe their very existence to the need of the elementary teacher for guidance and sympathy with his work not found in the college. There would have been no room for them had the colleges been interested in teaching how to teach boys and girls, much more scholarly work would have been done by the would-be teachers and instructors and sounder principles would no doubt have been evolved. The primary education in our country is excellent. The Mosely Commission claimed it far exceeded anything abroad and the development of the primary pupil here to be much in advance of anything they could offer. The plans and devices of the superintendents were pronounced to be marvelous in their completeness, but the commission reported at home that the grammar and high school pupils here were behind the English pupil of the same age. This can only be explained in one way—our teachers did inferior work. In putting the elaborate schemes into execution they did not make them a part either of their own or the pupils' experience and while the result was showy, it was not lasting. It is not so much superintendents and principals and text books that we need but teachers who meditate and encourage meditation and originality. The inspiration of a great teacher is more than fine buildings or exquisitely worked out courses of study and machine-like plans for administering them. Let us have men and women of power for teachers. The good work of a teacher does not show immediately. The charlatan and the opportunist will attract more attention than the teacher. This is true even in the "professions" in which classification we are not yet admitted by common consent and as a result of our own weaknesses.

Much of the early mathematics consisted of a crude arithmetic for practical use on one side and pure puzzles for show purposes on the other. This tendency gave way to the idea that mathematics was always an exhaustive development of a particular subject for its own sake and its own method. There has been for some time a decided tendency toward unification, not merely a gathering together of the forces for a new series of separate exploits but a realization of the interrelations, one to one correspondences, even multiple isomorphisms, of subjects heretofore kept in watertight compartments hermetically sealed. We are met in such associations as this for mutual advice and assistance, and there is not one of us so wise that he cannot learn of the least of those among us.

It has been well said that the purposes of teaching arithmetic are the following: (1) Drill for accuracy and speed in computation. (2) Interest in nature. (3) Acquaintance with the mathematical type of thought. (4) Preparation for further study. If these are properly expressed they will have extension to the so-called higher branches of the subject.

We will probably agree that there are in general three stages in most problems in mathematics. First, a translation of the known existing data into mathematical symbolism more perspicuous to the eye. Second, manipulation of this symbolism, *according to laws fixed by reason or custom, keeping in mind all the time the one to one correspondence between the actual and our ideal problem. And lastly, a retranslation from the symbolism to ordinary language and a verification of the result with known fact.*

It has been the neglect of the first and last of these stages and a partial neglect of the second that has made it possible for our ranks to be recruited at times from worn out preachers who have lost their voice and people not strong enough for other subjects. Some of our cotemporaries even now believe that any one can teach mathematics as every problem is either all right or all wrong and there can be but one answer and that one always in the book.

My uncle tells me that he once attended a dames school kept by two Quaker ladies of slender means, which slender means and the attitude in which the teaching profession was held were the sole reasons why they kept school. The highest mathematics taken was long division. He said he wrote down his divisor *out of the book,* then his dividend *out of the book,* then his quotient always *out of the book.* Then they made him multiply out that long tail below and why they made him do it he never knew, for they never told him and he believes they are placidly unconscious of it still.

I wonder how many of us as children wondered why in multiplication by a number of two or more digits we wrote the second set of figures one place to the left. I am sure it was long a puzzle to me, and my little boy came home from school the other day troubled over the same old question and was happy when I told him why.

Charles Dickens appreciated the radix method when he said in " Bleak House " that a certain man " was to his wife what zero is to nine in ninety—something with her and nothing without her." Place in mathematics as in the world is relative. The only real difference between the bank account of a teacher and John D. Rockefeller is where he has the power to place the decimal point.

Our ancestors like ourselves at a more or less remote date from the present spent many hours in earnest contemplation and admiration of their digits and finally evolved a method of enumeration which is now based on the original *ten* digits in our ordinary operations with number and from which has developed the power series of more general or algebraic number. There is no virtue in the radix *ten,* to make a poor pun it just happened to be *at hand.* The number 392 is $3(10)^2 + 9(10) + 2$. Whether the whole accidental and intuitive system to which we have referred is unduly cumbersome and as my friend and constant source of inspiration, Dr. Schwatt, has intimated may be illuminated or replaced when we understand the relations that exist between certain varieties of incommensurable numbers such as π and ϵ and others doubtless belonging to the same or different systems is a question for further study. Note $\epsilon = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \ldots$ where the radix does not occur to a power but appears as a product of a sequence. Certainly our operations in mathematics will bear inspection and cannot in the spirit of the subject be taken entirely on faith.

We all hear it stated and the world in general believes that the mathematics is a sort of Dunderbach's sausage-making machine. An eminent professor of philosophy said recently that the advantage of the calculus was that all you had to do was to turn the crank and your answer came out ready to use. The modern psychologist inveighs against us in that our reasoning is in a particular narrow line and states that reasoning in a particular line will not necessarily benefit reasoning in general. The schools say that the best hours of the day and more than one fifth of the time during the first ten years of school life is spent on mathematics with very meagre practical results and little of culture value. Who is responsible for this attitude?

If I mistake not there is a feeling among teachers of mathematics that there is room for improvement in methods and sequence of subject matter.

Most of the literature at present is of the destructive, rather than constructive type. Much of it for a time must be inspirational and of a sort to encourage supplementing the present type of texts and the gradual omission of useless or obsolete topics. This work must necessarily be slow and must come from the teachers for no publisher would dare cut out of his book all of the useless and nonsensical ideas therein. Even if it were possible to write a perfect text there would still be the individuality of the teacher to consider. It is a well established fact that a fair system well understood and taught will lead to better results than an excellent system in the hands of an unsympathetic instructor.

It is my purpose to discuss a certain type of mathematical thought because of its relations to practical work, accuracy and speed in computation, and as an aid to the understanding of the higher mathematics. I will make two statements:

1. *Every number we use in actual life is an average number.*
2. *The operations of mathematics will bear watching.*

Our judgments are based on the average of the sum total of our experiences. When we wish to measure any length accurately, we repeat the process several times and take the average of the several different estimates we have made as the most probable value. We can estimate a length to the nearest inch, or to the nearest sixteenth of an inch or to the nearest thousandth part of an inch, you measure to any degree of accuracy you may desire and I will obtain a closer approximation, and then you can come closer still with a more finely divided measure.

Here is the germ of the infinite series. The true length is the limit of the sum of this series. The difference between the true measure and your result is an infinitesimal. Your measurement is not exact. It is within the nearest artificial unit you may choose. The quantity you are measuring is not commensurable with your measure. Many a boy who is set to measuring a table as his first lesson in physical measurements presents a suspiciously uniform set of measurements and is sent back to do his work again. Many a surveying class goes back to the field to find that the angles of a triangle never sum up exactly to 180°. Now are any of these ideas impossible of comprehension to the average school boy. They are fundamental conceptions needed in higher mathematics and in practical work, and yet they are absolutely new to ninety-nine out of every hundred freshmen in college and engineering schools.

And now a word about the statement that the operations of mathematics will bear watching. Suppose we want the product of the two numbers 258.73 and 24.568 correct to five significant figures (note I did not say decimal places). See Problem 1. Placing units under units so as not to be troubled about place, we will multiply the numbers in the ordinary way except to change the order of the products as we do in algebra and obtain as our product, 6356.47864. Now will it not occur to the thinking student that we have done an unnecessary amount of work in order to obtain five figures? Would not the process in Problem 2 have obtained the result more expeditiously?

PROBLEM 1.	PROBLEM 2.
258.73...	258.73...
24.568..	24.568..
5174.6....	5174.6....
1034.92...	1034.9....
129.365..	129.4....
15.5238.	15.5....
2.06984	2.1....
6356.47864	6356.5
6356.5	

The operation might be conducted in the following manner. Two tens by three hundredths gives six tenths, &c., for the first product. Now cross off the two tens that you have used in multiplying and you can do the same with the three hundredths for units times hundredths gives hundreths. You will however need to note that the product would have been four units by three hundredths gives twelve hundredths, which is nearest to one tenth and with the twenty-eight tenths next obtained make twenty-nine tenths and so on each time crossing off a figure from multiplier and multiplicand until you are forced to stop because you have no longer numbers of which to take the product. This process is evidently much shorter than the one in ordinary use for exact numbers. Now let us suppose these are not exact numbers but numbers such as we obtain from measurements, one of the numbers being given to the nearest hundredth of the unit and the other to the nearest thousandth part. (There are approximate integers as well.)

Now the eight in the thousandth place (Problem 1) is not quite correct neither is the three in the hundredths place, hence the four in the hundred thousandth place is even more doubtful, and since all the dots represent missing unknown digits our first problem is to add to a very doubtful number four other numbers of which we are entirely ignorant. Certainly the chances are nine to one against the sum being a number ending in four and as De Morgan says we have even greater doubt about some of the others. In fact we do not know very much if anything about any unfinished column, and we certainly can place no dependence on any more of our work than we used in our abbreviated process (Problem 2). Even the last figure here will bear close scrutiny and we will find in general that the result is correct to one significant figure less than we obtained from our observation. In fact in long and complicated examples we had better correct to two less figures or make an examination of the limits of error which is easily done in any particular example. In this example you will notice that we are not absolutely sure of anything beyond the corrected tens place, and that our result is about an average of the possible values. The problem not only shortens up the work but calls attention to the accuracy of the result and prevents many errors.

Problem 3.

$$\begin{array}{rr}
 & \underline{5.34831} \\
 & 3.142)\,16.80437921 \\
 & \underline{15\,710} \\
 & 1\,0943 \\
 & \underline{9426} \\
 & 15177 \\
 & \underline{12568} \\
\underline{2610} & 26099 \\
\underline{2514} & \underline{25136} \\
\underline{96} & 9632 \\
\underline{94} & \underline{9426} \\
\underline{2} & \underline{2061}
\end{array}$$

Problem 4.

$$\begin{array}{r}
\underline{5.348} \\
3.142)\,16.80437921 \\
\underline{15\,710} \\
1\,094 \\
\underline{943} \\
151 \\
\underline{126} \\
25 \\
\underline{25}
\end{array}$$

Note also the problem in division (see Problem 3) again supposing the numbers exact. It is convenient with young pupils to have the decimal point after the first figure of the divisor in order to make the division more conveniently but this is evidently not necessary. *We will note all through that there is absolutely no difference in the problem whether there is a decimal point expressed or understood. There should be no difference in the treatment of decimals and whole numbers.* The point merely changes the size of the unit. As soon as the remainder (921) is less than the divisor (3142), it is convenient to stop bringing down more figures and obtain the remaining digits of the quotient by continually contracting the divisor and dropping the rest of the dividend. Thus cutting off the 2 we see 314 is contained in the 2610 eight times, and since 8 times 2 is 16 we have

2 to "carry" from the cut off digit. Similarly cutting off successive digits the operation is as on the left.

The process is the same if only the dividend be approximate, but if the divisor is approximate the last figures in the dividend will be useless and we will be reasonably sure of only four figures in this case. There is evidently no need to have more digits in the divisor than in the quotient (see Problem 4). The extreme folly of annexing ciphers in such cases as this and in any case of supposing that a calculation of any kind could give results more accurate than our data or indeed as accurate except by accident is self evident. A certain celebrated chemist would not have formed an erroneous theory, nor would a prominent psychologist just recently have advanced an equally fallacious statement in one of our scientific journals if they had understood the operations they were so thoughtlessly using. I once asked one hundred public school teachers to multiply $\pi = 3.14159$ by the $\sqrt{2}$ and give result correct to two places of decimals and got one correct result. But what surprised me most was the fact that many elderly carpenters and machinists were able to do the problem. The carpenter in obtaining the square root of 2 would say it is more than 1. He would divide 1 into 2 and getting two would take the average $1\frac{1}{2}$, then he would divide $1\frac{1}{2}$ into 2 and get $1\frac{1}{3}$ and again take the average until he had the required accuracy.

It is absolutely impossible to write the $\sqrt{2}$ in our ordinary system of decimal notation and its conception requires an extension of our idea of the square root just as new to the ordinary pupil as the idea of fourth dimensional or *n* dimensional geometry is to a student of Euclidean geometry.

We can draw the diagonal of a square of unit side and thereby construct the geometric correspondent to our extended notion of the $\sqrt{2}$. But it is wise to keep in mind in such problems of correspondence that though a point on a line may be made to correspond to every real number the converse requires examination. I will here remark that it is probable that this idea of correspondence would save the secondary teacher much trouble in constructing graphs of quadratic equations on paper ruled in squares. For example, if the side of one square be taken as the unit, the diagonal is the $\sqrt{2}$. A rectangle, $\sqrt{2}$ by 1, has $\sqrt{3}$ for its diagonal. The $\sqrt{5}$ is obtained from the diagonal of a rectangle, two units by one unit. For the $\sqrt{6}$ we may take the diagonal of the rectangle, 2 by $\sqrt{2}$. The $\sqrt{3}$ might have been taken by constructing a right triangle with base 1 and hypotenuse 2. The altitude evidently being the $\sqrt{3}$. The $\sqrt{8}=2\sqrt{2}$; set compass $=\sqrt{2}$ and lay off length twice. Take for instance,

$$y=(3+2\sqrt{5})/7.$$

We would construct $\sqrt{5}$ as above. Set the compass to this length and step it off twice in the required direction, then set to unit length and step off 3 in addition. The ordinary geometrical method of dividing a line into equal parts will now suffice to get the seventh part, but it will be just as accurate for our purpose to obtain it by trial with the dividers as the ordinary draughtsman does. The necessity of rationalizing the denominator is evident both in the numerical process and the geometric counterpart.

For extensions of the approximation process to other forms of computation abbreviated methods of computation with exact numbers such as the Italian method of division and the like, I will refer the interested teacher to such books as Langley's "Computation," Longman, Green & Co., New York; Skinner's "Approximate Computation," Henry Holt & Co., New York; De Morgan's "Elements of Arithmetic" (this last probably obtainable only in libraries); or any good English arithmetic. *I would like to emphasize the fact that I referred to the interested teacher, keeping in mind that the teacher's standpoint is different from the pupils and that there are many things that the teacher can introduce, should introduce and must introduce at the psychological moment, without reference to any set course of study or supervision of principal or superintendent.* Our curriculum to-day is clogged with courses which were meant to be parts of several main subjects. Intense specialization and over zealousness to exploit some single idea for advertising or other purposes has led us many times to cut asunder things that have no life apart.

In ordinary simple measurements we have an opportunity to find examples of mathematical thought that will enable our boys and girls not only to realize that mathematics is not a mere juggle of symbols by the turning of a crank, but a science full of the very beauties our psychologists say it lacks.

The curious questioning mind of the child, contemplative as it is even in the cradle, will easily become what we call the inquiring mind—the mind of the investigator if it is not dwarfed by system in education. Meditation by teacher and pupil as well as supervisor is imperative. The simple, toilful inductive methods of the investigator are not often seen in the glittering array of logic of the finished product. To primitive instinct to conceal the treasure house and from a desire to present the broad view gained only after tedious search, added to a wish to save time we can trace the ordinary deductive form of presentation. This style is only possible after having a knowledge of the result. Can we not let the child and the ordinary layman into the secret and take away some of the trappings and illusions and teach more real power instead of so many cyclopedic statements. At least we can show him that the carrying out of mathematical operations beyond certain limits is illogical and useless and absurd and untruthful and that even as it is desirable for a speaker or writer to know when to stop—so even more so is it for the computer.

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